From kinetic scales to fluid scales in space plasmas: quantities requiring a multipoint information

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The understanding of space plasma dynamics requires the investigation of processes covering an extremely wide range of scales ($> 9 - 10$ orders of magnitude).

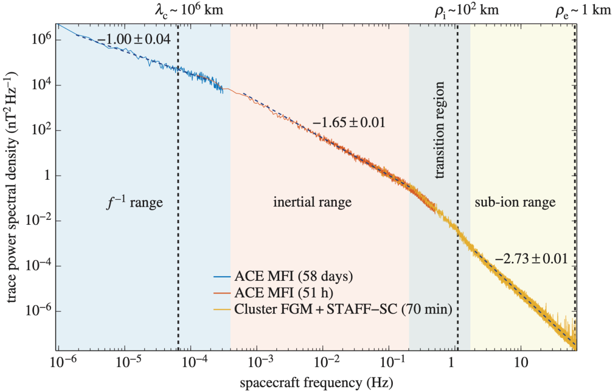


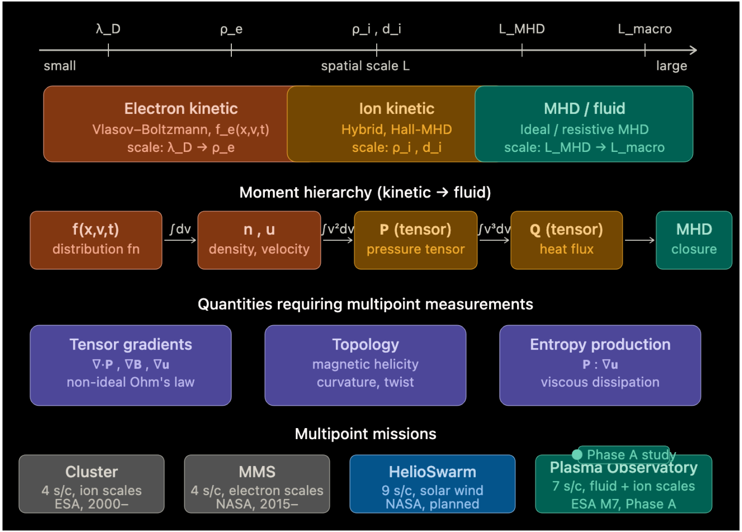
Figure: from Kiyani et al., Phil. Trans. R. Soc. A, 2015

This wide range of scales extends from *kinetic* (Debye length, gyroradius) up to *fluid* (MHD, large-scale structures), involving the investigation of both *local and non-local quantities and processes*.

In particular, the study of *non-local quantities* (as for instance coarse-grained magnetic and velocity field helicity, topological features, ...) as well as some local quantities (currents, $\nabla \mathbf{B}$, $\nabla \mathbf{u}$, ...) cannot be investigated using single-point measurements.

That requires a multipoint approach.

Introduction: 2



Klimontovich Equation: 1

Let's start our description of plasma evolution in terms of an ensemble of individual particles.

As a first step we can introduce a density function, $N_s(\mathbf{r}x\mathbf{v};t)$, in the Boltzmann's single-particle phase space, $(\mathbf{x},\mathbf{v})$, for each species s, which provides the exact number of particles with velocity $\mathbf{v}$ at the location $\mathbf{r}$;

$$\boxed{N_s(\mathbf{x},\mathbf{v},t)=\sum_{i=1}^{N_0^s}\delta(\mathbf{x}-\mathbf{x}_i(t))\,\delta(\mathbf{v}-\mathbf{v}_i(t))\,.} \tag{1}$$

where $(\mathbf{x}_i,\mathbf{v}_i)$ are the Lagrangian coordinates of the particles of the species s and $(\mathbf{x},\mathbf{v})$ the coordinates of the 6d phase space.

Taking into account the single particle motion equation,

$$m_s\dot{\mathbf{v}}_i(t)=q_s\mathbf{E}_m[\mathbf{x}_i(t),t]+\frac{q_s}{c}\mathbf{v}_i(t)\times\mathbf{B}_m[\mathbf{x}_i(t),t], \tag{2}$$

where m indicates the microscopic (local) fields, given by the Maxwell's equations, ones gets the Klimontovich Equation.

$$\boxed{\frac{\partial}{\partial t}N_s+\mathbf{v}\cdot\nabla_x N_s+\frac{q_s}{m_s}\left[\mathbf{E}_m+\frac{1}{c}\mathbf{v}\times\mathbf{B}_m\right]\cdot\nabla_v N_s=0} \tag{3}$$

Klimontovich Equation: 2

Klimontovich Equation is exact. This is a continuity equation (analogous to the Liouville Equation) asserting that the N_s is constant along the trajectory in the phase space, i.e.,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{traj} \cdot \nabla_x + \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{traj} \cdot \nabla_v \quad \longrightarrow \quad \frac{D}{Dt} N_s = 0$$

Here, $\mathbf{E}_m$ and $\mathbf{B}_m$ are the **total microscopic fields generated by all particles**, including the particle at $\mathbf{x}_i$ itself (minus self-interaction), making it coupled to the exact positions of all N particles. It is therefore completely intractable as written.

The Klimontovich equation is practically not solvable, being equivalent to solve all the particles' motion.

Furthermore, they contain all the information that may be too much to describe the plasma evolution at the investigated spatial and time scales.

A way to overcome this problem is to average the N_s over an ensemble of replicas of the original system introducing the phase-space distribution function f_s,

$$\boxed{f_s\left(\mathbf{x}, \mathbf{v}, t\right) = \langle N_s\left(\mathbf{x}, \mathbf{v}, t\right)\rangle,} \tag{4}$$

This operation, which is equivalent to move from a singular function to a continuous function, has a non-trivial drawback.

It is equivalent to a breaking of conservation, indeed we are moving from a numerable discrete set to a continuous one [Ageno, 1995].

Using the ensemble average, we can express the Klimontovich density function as,

$$N_s(\mathbf{x}, \mathbf{v}, t) = f_s(\mathbf{x}, \mathbf{v}, t) + \delta N_s(\mathbf{x}, \mathbf{v}, t) \tag{5}$$

i.e., we can decompose N_s into a mean (ensemble average) plus a fluctuation, with

$$\langle \delta N_s(\mathbf{x}, \mathbf{v}, t) \rangle = 0. \tag{6}$$

The same can be done for the fields, $\mathbf{E}_m$ and $\mathbf{B}_m$, so that substituting in the Klimontovich equation we get

$$\boxed{\partial_t f_s + \mathbf{v} \cdot \nabla_x f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \nabla_v f_s = -\frac{q_s}{m_s}\left\langle \left(\delta\mathbf{E} + \frac{\mathbf{v}}{c} \times \delta\mathbf{B}\right) \cdot \nabla_v \delta N_s \right\rangle} \tag{7}$$

This is Kinetic Equation for plasmas.

On the left hand we have the mean fields and the distribution function, i.e., smooth quantities, while in the right hand we have the fast variable terms (fluctuations, collisions).

More specifically, the right-hand side is the correlation term $\mathcal{C}_s$. It involves the two-point correlation, which itself satisfies an equation involving three-point correlations, and so on. This is the origin of the Born-Bogoliubov-Green-Kirkwood-Yvonne (BBGKY) hierarchy.

The BBGKY hierarchy implies the solution for a hierarchy of equations associated with the n-particle distribution functions,

$$\begin{aligned} f_s^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) &= \langle N_s(\mathbf{x}_1, \mathbf{v}_1, t) \rangle \qquad (8) \\ f_{ss'}^{(2)}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) &= \langle N_s(\mathbf{x}_1, \mathbf{v}_1, t) N_{s'}(\mathbf{x}_2, \mathbf{v}_2, t) \rangle \qquad (9) \\ \ldots\ldots\ldots &= \ldots\ldots\ldots\ldots \end{aligned}$$

The evolution of $f^{(1)}$ depends on $f^{(2)}$ the evolution of $f^{(2)}$ depends on $f^{(3)}$, and so on.

To close the hierarchy at the one-particle level, one writes the two-particle distribution as:

$$f_{ss'}^{(2)}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f_s^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) f_{s'}^{(1)}(\mathbf{x}_2, \mathbf{v}_2, t) + g_{ss'}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) \qquad (10)$$

where $g_{ss'}$ is the two-particle correlation function, measuring deviations from statistical independence.

The BBGKY truncation consists of neglecting $g_{ss'}$ or treating it perturbatively. The physical small parameter controlling this is:

$$\epsilon = \frac{1}{n\lambda_D^3} = \frac{1}{\Lambda} \qquad (11)$$

where $\Lambda = n\lambda_D^3$ is the plasma parameter. For $\Lambda \gg 1$ (which holds in virtually all space plasmas), binary correlations are weak and the truncation is justified.

The Moment Hierarchy: 1

In certain situations, the kinetic description provides a level of detail that is not accessible through observation, as it is in fact a statistical-microscopic description of the dynamics of a plasma.

The fluid description, which is based on the evolution equations for the velocity moments of the distribution function, provides an alternative description for the plasma dynamics.

In general, the average quantities of interest are those functions Φ that are multilinear products of the components of velocity $\mathbf{v}$, namely

$$\Phi = v_i v_j \cdots v_k, \tag{12}$$

From the Boltzmann equation we get for the generic moment Φ the following evolution equation,

$$\boxed{\partial_t \left(n_\alpha \langle \Phi \rangle_\alpha\right) + \nabla_x \cdot \left(n_\alpha \langle \Phi \mathbf{v} \rangle_\alpha\right) - \frac{q_\alpha}{m_\alpha} n_\alpha \langle \nabla_v \Phi \cdot \left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \rangle_\alpha = n_\alpha \langle C_\alpha \Phi \rangle_\alpha} \tag{13}$$

In particular let us consider the following moments:

$$\Phi = 1, \qquad \Phi = m_\alpha \mathbf{v}, \qquad \Phi = \frac{1}{2} m_\alpha v^2 \tag{14}$$

This will allow us to derive, respectively, *the equations for the density, momentum, and energy balances*.

- The density balance equation: $\Phi = 1$:

$$\boxed{\partial_t n_\alpha + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0} \quad (15)$$

- The moment balance equation: $\Phi = m_\alpha \mathbf{v}$:

$$\boxed{m_\alpha n_\alpha \frac{d}{dt}\mathbf{u}_\alpha = -\nabla p_\alpha - \nabla : \overline{\mathbf{\Pi}}_\alpha + q_\alpha n_\alpha \left(\mathbf{E} + \frac{1}{c}\mathbf{u}_\alpha \times \mathbf{B}\right) + m_\alpha n_\alpha \langle C_\alpha \mathbf{v} \rangle_\alpha} \quad (16)$$

where

$$\overline{\mathbf{P}}_\alpha = p_\alpha \overline{\mathbf{1}} + \overline{\mathbf{\Pi}}_\alpha \quad (17)$$

being p_α the scalar pressure and $\overline{\mathbf{\Pi}}_\alpha$ the *viscous stress tensor*.

- The energy balance equation, $\Phi = \frac{1}{2} m_\alpha v^2$:

$$\boxed{\partial_t n_\alpha K_\alpha + \nabla \cdot \mathbf{G}_\alpha = q_\alpha n_\alpha \mathbf{u}_\alpha \cdot \mathbf{E} + \frac{1}{2} m_\alpha n_\alpha \langle C_\alpha v^2 \rangle_\alpha} \quad (18)$$

where

$$K_\alpha = \frac{3}{2}\kappa_B T_\alpha + \frac{1}{2} m_\alpha u_\alpha^2 \quad (19)$$

and

$$\mathbf{G}_\alpha = \mathbf{q}_\alpha + \overline{\mathbf{P}}_\alpha : \mathbf{u}_\alpha + \frac{3}{2} n_\alpha \kappa_B T_\alpha \mathbf{u}_\alpha + \frac{1}{2} m_\alpha n_\alpha u_\alpha^2 \mathbf{u}_\alpha \quad (20)$$

The single fluid equations: 1

Considering a plasma consisting of two species e^- and ions (protons) and introducing the following quantities:

- the mass density $\rho(\mathbf{r}, t)$: $\rho(\mathbf{r}, t) = m_e n_e(\mathbf{r}, t) + m_i n_i(\mathbf{r}, t)$,
- the charge density $\rho_e(\mathbf{r}, t)$: $\rho_e(\mathbf{r}, t) = e(n_i - n_e)$,
- the current density $\mathbf{j}(\mathbf{r}, t)$: $\mathbf{j}(\mathbf{r}, t) = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$,
- the mass center velocity $\mathbf{u}(\mathbf{r}, t)$: $\mathbf{u}(\mathbf{r}, t) = \frac{1}{\rho} \left[m_e n_e \mathbf{u}_e(\mathbf{r}, t) + m_i n_i \mathbf{u}_i(\mathbf{r}, t) \right]$,

it is possible to construct a single fluid theory.

In detail we have the following set of equation:

- The continuity equation for the density,

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0} \tag{21}$$

- The conservation equation for the charge,

$$\boxed{\frac{\partial}{\partial t}\rho_e + \nabla \cdot \mathbf{j} = 0} \tag{22}$$

- The conservation equation for the momentum,

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla : (\rho \overline{\mathbf{uu}}) + \nabla : \overline{\mathbf{P}} = \rho_e \mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}} \tag{23}$$

- The energy balance equation:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}p\right) + \nabla \cdot \left[\frac{1}{2}\rho u^2 \mathbf{u} + \frac{3}{2}p\mathbf{u} + \overline{\mathbf{P}} : \mathbf{u} + \mathbf{q}\right] = \mathbf{j} \cdot \mathbf{E}} \tag{24}$$

- The generalized Ohm's law:

$$\boxed{\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B} = \frac{1}{nec}\mathbf{j} \times \mathbf{B} - \nabla : \left(\overline{\mathbf{P}}_e - \frac{m_e}{m_i}\overline{\mathbf{P}}_i\right) + \frac{m_e}{ne^2}\left[\frac{\partial}{\partial t}\mathbf{j} + \nabla : \left(\overline{\mathbf{uj}} + \overline{\mathbf{ju}}\right)\right] + \eta \mathbf{j}} \tag{25}$$

These equations contain the estimation of some quantities which are not possible to be estimated without strong assumption using a single point measurements, e.g.

$$\nabla : \overline{\mathbf{P}}, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad \nabla : \overline{\mathbf{uj}}, \quad ..... \tag{26}$$

The single fluid equations: 3

This system of equations must, of course, be combined with Maxwell's equations, specifically for $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{B}$ thus forming the complete set of equations for describing plasma as a fluid subject to electromagnetic effects.

In particular, for the MHD model, which is valid at the scales where the motion of the single constituents is not distinguishable, we have that the set of the equations coupled with the Maxwell's equations reduces to:

$$\begin{cases} \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} \\ \frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B} \\ \frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\mathbf{u}) = 0 \\ \rho\frac{d}{dt}\mathbf{u} = -\nabla p + \frac{1}{c}\mathbf{u} \times \mathbf{B} \\ \frac{d}{dt}\left(p\rho^{-\gamma}\right) = 0 \end{cases} \tag{27}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$.

Here, some quantities have been neglected and the plasma is considered collisionless.

An interesting quantity to be investigated in the case of space plasmas is the *entropy production rate* σ_s, which is a quantity associated to dissipation.

Moving from the *Boltzmann-Gibbs entropy*, i.e.,

$$S_s = -\kappa_B \int f_s \ln f_s d^3 x d^3 v \tag{28}$$

we can introduce the local entropy density,

$$\eta_s(\mathbf{x}, t) = -\kappa_B \int f_s \ln f_s d^3 v \tag{29}$$

which from the Boltzmann equation evolves according to the following equation,

$$\frac{\partial}{\partial t}\eta_s + \nabla \cdot \mathbf{J}_s^{\eta} = \sigma_s^{kin} \tag{30}$$

Here, $\mathbf{J}_s^{\eta}$ is the *kinetic entropy flux* and σ_s^{kin} the *kinetic entropy production rate*,

$$\boxed{\sigma_s^{kin} = -\kappa_B \int \mathcal{C}_s \ln f_s d^3 v \geq 0} \tag{31}$$

where $\mathcal{C}_s$ is the *collision operator*. The inequality follows from the *H-theorem*.

Entropy production: 2

In the collisionless (Vlasov) limit $\mathcal{C}_s = 0$ and thus *entropy is conserved*, i.e., $\sigma_s^{kin} = 0$.

However, observation in real space plasmas show *irreversible heating and dissipation* even in the situation/regimes where collisions are negligible.

A possible solution to this apparent paradox lies in phase-space mixing or filamentation. Indeed, the coarse grained distribution over a phase-space volume $\delta\Omega$ $\overline{f}_s(\mathbf{x}, \mathbf{v}, t) = \langle f_s \rangle_{\delta\Omega}$, produces a $\overline{\eta}_s \geq \eta_s$ (Jensen's inequality).

Moving towards the fluid entropy production, it can be derived the fluid entropy equation,

$$\rho T \frac{d}{dt} s = -\nabla \cdot \mathbf{q} - \overline{\mathbf{\Pi}} : \nabla \mathbf{u} + \frac{\eta}{\mu_0} \mid \mathbf{j} \mid^2 + \ldots.. \tag{32}$$

so that the entropy production rate density is

$$\sigma = -\frac{1}{T}\overline{\mathbf{\Pi}} : \nabla \mathbf{u} - \frac{1}{T^2}\mathbf{q} \cdot \nabla T + \frac{\eta}{\mu_0} \mid \mathbf{j} \mid^2 = \left[\sum_i X_i J_i\right] \geq 0 \tag{33}$$

Again the estimation of some of the terms of the entropy production rate requires multiipoint measurements.

In detail, we need at least 4 satellites to estimate the terms: $-\overline{\mathbf{\Pi}} : \nabla \mathbf{u}$ (also known as *PiD*), the ∇T and $\mathbf{j} = \nabla \times \mathbf{B}$.

Entropy production: 3

The stress tensor $\overline{\mathbf{\Pi}}$ can be decomposed into two parts:

- the strain rate tensor (symmetric part): $S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$;
- the vorticity tensor (antisymmetric part): $W_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j)$.

Here, the symmetric part can be written as

$$S_{ij} = \left(S_{ij} - \frac{1}{3}\theta\delta_{ij}\right) + \frac{1}{3}\theta\delta_{ij} \tag{34}$$

where the first term in the right is the **traceless strain**, $\tilde{S}_{ij}$, and the second one the **dilatation term** with $\theta = \nabla \cdot \mathbf{u}$.

In nonequilibrium thermodynamics, **linear phenomenological relations** allow to relate the fluxes with forces of the same tensorial character, so that we can write

$$\overline{\mathbf{\Pi}} = -2\mu\tilde{S}_{ij} - 2\zeta\theta\delta_{ij} \tag{35}$$

with $\mu \geq 0$ **dynamic (shear) viscosity** and $\zeta \geq 0$ **bulk viscosity**.

From this approximation, which is valid in the case of symmetric systems, we obtain for the viscous term in the entropy production rate the following result

$$\sigma_{visc} = \frac{2\mu}{T}\tilde{S}_{ij}\tilde{S}_{ij} + \frac{\zeta}{T}\theta^2 \tag{36}$$

The situation is more complex when symmetry fails, as it occurs in magnetized plasmas implying a break down of separation between dissipative strain and non-dissipative vorticity at kinetic scales.

Morphology: 1

The knowledge of quantities as velocity and magnetic field field tensors, $\overline{\mathbf{A}} = \nabla \mathbf{u}$ and $\overline{\mathbf{Z}} = \nabla \mathbf{B}$, can also provide information on the morphological features of the streamlines and the magnetic field lines (Vieillefosse 1984; Cantwell 1992, 1993).

The early theoretical studies on the properties and the evolution of the gradient tensors dates back to the 80s–90s in the framework of fluidodynamics (Vieillefosse 1982, 1984; Cantwell 1992).

As clearly shown by the early studies of Cantwell and co-Authors, the knowledge of velocity gradient tensors, defined as $A_{ij} = \partial_j U_i$, may provide information on the small-scale structures and dynamics such as the occurrence of the nonlinear self-stretching, which is responsible for the generation of small-scale large fluctuations in turbulent fluids.

Moving from the definition of the gradient tensors of a field, it is possible to introduce a set of geometrical scalar invariants under SO(3) that can be defined starting by the characteristic equation for the eigenvalues (λ) of the gradient tensor, i.e.,

$$|| \overline{\mathbf{M}} - \lambda \overline{\mathbf{I}} || = \lambda^3 + \mathcal{A}\lambda^2 + \mathcal{B}\lambda + \mathcal{C} = 0 \tag{37}$$

where $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are invariant quantities, that according to Cayley-Hamilton theorem, are conserved under the SO(3) group. The three invariants are defined as

$$\mathcal{A} = -\mathrm{Tr}(\overline{\mathbf{M}}), \quad \mathcal{B} = \frac{1}{2}\left[\mathcal{A}^2 - \mathrm{Tr}(\overline{\mathbf{M}}^2)\right], \quad \mathcal{C} = -\mathrm{Det}(\overline{\mathbf{M}}) \tag{38}$$

or equivalently

$$\mathcal{A} = -\sum_i \lambda_i, \quad \mathcal{B} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad \mathcal{C} = -\lambda_1\lambda_2\lambda_3 \tag{39}$$

Morphology: 2

The roots of the characteristic polynomial provide information on the morphology of the field lines.

Assuming $\lambda_1 \in \mathbb{R}$, the remaining pair of eigenvalues can be real or complex, so that the field lines may be associated with *strain* or *vortical configurations*, respectively.

Moreover the **sign of the real part** $\mathrm{Re}\lambda_{2,3}$ is associated with stable ($\mathrm{Re}\lambda_{2,3} < 0$) or unstable ($\mathrm{Re}\lambda_{2,3} > 0$) local morphologies of the field lines. In addition, the sign of the first eigenvalues λ_1 determines if the morphology is compressing, $\lambda_1 < 0$, or stretching, $\lambda_1 > 0$.

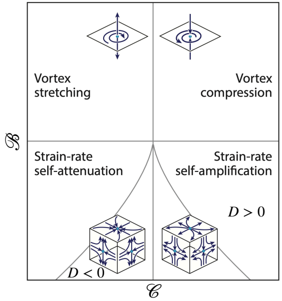


Figure: Adapted from Johnson & Wilczek, 2024

For solenoidal fields (incompressible flows or magnetic field, $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{B} = 0$), we have that $\mathcal{A} = 0$.

In this case, it it possible to classify the different morphologies by simply considering the two invariants $\mathcal{B}$ and $\mathcal{C}$ (Chong et al. 1990; Cantwell 1992; Asimov 1993; Meneveau 2011). In detail, the $(\mathcal{C}, \mathcal{B})$ plane can be divided into four sectors using the $\mathcal{C} = 0$ axis and the line $D = 0$, associated with the discriminant, i.e.,

$$D = 27\mathcal{C}^2 + 4\mathcal{B}^3 = 0 \tag{40}$$

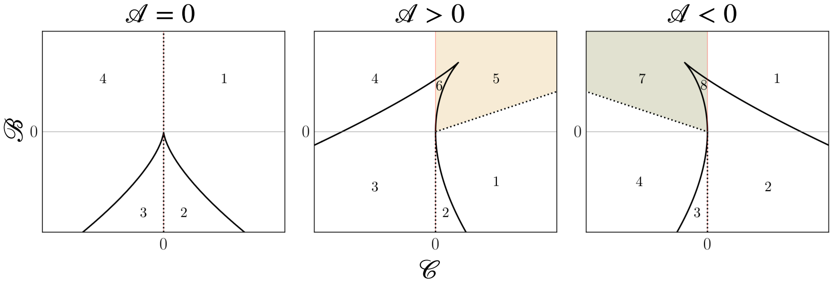
1. $D > 0$ & $\mathcal{C} > 0$: ingoing spiral saddle (ISS)
2. $D > 0$ & $\mathcal{C} < 0$: outgoing spiral saddle (OSS)
3. $D < 0$ & $\mathcal{C} < 0$: tube-like structures(TLS)
4. $D < 0$ & $\mathcal{C} > 0$: sheet-like structures (SLS)

In the case of compressible flows $\mathcal{A} \neq 0$ different scenarios become possible, as local volumetric expansion or contraction along all directions (Chong et al., 1990; Suman & Girimaji 2010; Wang & Lu 2012; Vaghefi & Madnia 2015; Zheng et al. 2022).

If $\mathcal{A} > 0$ then unstable transverse topologies are associated with a compression stronger than on the principal direction, while, if $\mathcal{A} < 0$ stable transverse topologies are always coupled with a stretching rate larger than $\lambda_2 + \lambda_3$ along the principal direction.

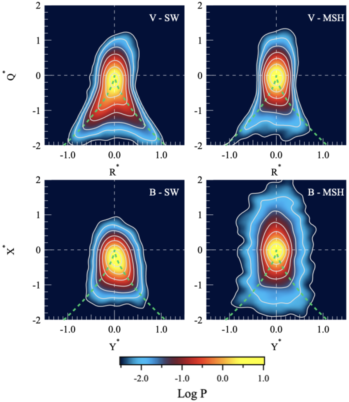
The discriminant surface, separating real from complex solutions, is given by the line

$$D = 27\mathcal{C}^2 + (4\mathcal{A}^3 - 18\mathcal{A}\mathcal{B})\mathcal{C} + (\mathcal{B}^3 + \mathcal{A}^2\mathcal{B}^2) = 0 \tag{41}$$



Adapted from Benella et al., 2026

The investigation of the joint-distributions of gradient tensor invariants allows to characterize the morphology of the streamlines and magnetic field lines in different space plasmas regions.



From Quattrociocchi et al., 2025

The field gradient tensor $\overline{\mathbf{M}}$ can be always decomposed into two different tensors representing its symmetric part, $\overline{\mathbf{K}}$, and anti-symmetric one, $\overline{\mathbf{J}}$,

$$\overline{\mathbf{M}} = \overline{\mathbf{K}} + \overline{\mathbf{J}} \tag{42}$$

These two tensors are associated with different field features. For the velocity field gradient tensor $\overline{\mathbf{A}} = \nabla \mathbf{u}$ we have that the symmetric part,

$$S_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right) \tag{43}$$

represents the strain tensor, while the anti-symmetric part,

$$\Omega_{ij} = \frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right) = \epsilon_{ijk}\omega_k \tag{44}$$

is associated to the flow vorticity $\omega = \nabla \times \mathbf{u}$.

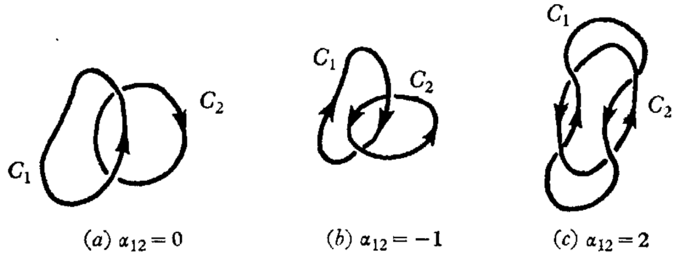
Analogously, for the magnetic field gradient tensor the symmetric part is the strain rate and the anti-symmetric part is associated with the current density $\mathbf{j} = \nabla \times \mathbf{B}$.

Given a vector field **X**, as for instance the magnetic field **B** and/or the velocity field **u**, confined to a domain $\mathcal{D}$ the integral

$$\mathcal{H} = \int_{\mathcal{D}} \mathbf{X} \cdot (\nabla \times \mathbf{X})\, dV \tag{45}$$

is called helicity. This integral was described as the asymptotic Hopf invariant by Arnold (1974) (Moffatt, 1969; Moffatt & Tsinober, 1992; Ricca, 2025).

The asymptotic Hopf invariant is an invariant of a divergence-free vector field on a three-dimensional manifold with given volume element, invariant under the group of volume-preserving diffeomorphisms, and describes the *helicity*.



From Moffatt, 1969

Helicity provides a measure of the topological complexity of the vector field, being a measure of the *linking and knottedness* of the field lines.

The linking is preserved in the case of ideal fluids when the dissipative effects are null.

Changing of topology and/or linking features requires non-ideal processes to take place.

This invariant appears naturally in ideal MHD and fluid mechanics,

$$H = \int_{\mathcal{D}} \mathbf{A} \cdot \mathbf{B} dV, \qquad I = \int_{\mathcal{D}} \mathbf{u} \cdot \omega dV \tag{46}$$

as the magnetic helicity H and the kinetic helicity I.

Following Moffatt (1969) it is possible to demonstrate that for an *inviscid fluid with uniform density* this quantity corresponds to the linkage of vortex filaments.

$$I = \sum_{i,j} \alpha_{i,j} \kappa k_i \kappa_j \tag{47}$$

where $\alpha_{i,j}$ is the winding number and κ_l is the vortex strength (or circulation), defined as

$$\kappa = \oint_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} = \int_{\mathcal{S}_{\mathcal{C}}} \omega \cdot d\mathbf{S}. \tag{48}$$

The same is valid in the case of the magnetic helicity H, that in the case of linked flux tubes can be written as

$$H = \sum_{i,j} L_{i,j} \Phi_i \Phi_j \tag{49}$$

where Φ_l is the magnetic flux of each tube and $L_{i,j}$ is the linking number between the tubes.

Topology: 3

Using multipoint observations from satellites it is possible to estimate coarse-grained helicities and/or magnetic h_m and kinetic h_k helicity densities.

For instance, reminding that the magnetic helicity density is $h_m(\mathbf{x}) = \mathbf{A}(x) \cdot \mathbf{B}(x)$, and that in the Coulomb gauge, $\nabla \cdot \mathbf{A}(x) = 0$, then

$$\mathbf{A}(x) = \frac{4\pi}{c} \int \frac{\nabla' \times \mathbf{B}(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} d^3x' = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} d^3x' \tag{50}$$

we get

$$h_m(x) = \left[\frac{1}{c} \int \frac{\mathbf{j}(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} d^3x' \right] \cdot \mathbf{B}(\mathbf{x}) \tag{51}$$

It is analogously possible to define the kinetic helicity density h_k as

$$h_k(\mathbf{x}) = \mathbf{u}(\mathbf{x}) \cdot [\nabla \times \mathbf{u}(\mathbf{x})] = \mathbf{u}(\mathbf{x}) \cdot \omega(\mathbf{x}). \tag{52}$$

An important issue is that in noncollisional (ideal MHD) the magnetic helicity is conserved (Woijter Theorem), the kinetic helicity may evolve,

$$\frac{\partial}{\partial t} h_k(\mathbf{x}) + \nabla \cdot \mathbf{F}_k = S_k - D_k \tag{53}$$

where $\mathbf{F}_k$ is the kinetic helicity flux, $D_k \simeq 0$ is the kinetic helicity dissipation and S_k is a kinetic helicity source,

$$S_k = \frac{1}{\rho} \left[\omega \cdot (\mathbf{j} \times \mathbf{B}) + \mathbf{u} \cdot \nabla \times (\mathbf{j} \times \mathbf{B}) \right]. \tag{54}$$

Summary

Here, we have just traced a simple way to some relevant quantities requiring multipoint observations to be computed and/or resolved.

All these quantities are m-dimensional tensors which appear in the governing equations of plasma physics from Klimontovich to MHD.

These equations are field equations relating quantities to their spatial derivatives.

Single-point measurements give the quantities.

Multipoint measurements give the derivatives.

The physics lives in the derivatives.